Practice Exam 1

1. An estimator $\hat{\theta}$ for θ has the following properties:

$$\mathbf{E}[\hat{\theta}] = 4$$
$$\mathrm{Var}(\hat{\theta}) = 20$$

If $\theta = 6$, calculate the bias of $\hat{\theta}^2$ as an estimator for θ^2.

A. Less than -3

B. At least -3, but less than -1

C. At least -1, but less than 1

D. At least 1, but less than 3

E. At least 3

2. For an auto collision coverage, average claim size (including the deductible) Y is related to the deductible X by the formula:

$$Y = a + bX + \varepsilon$$

Data for average claim sizes (including the deductible) is given in the following table:

Deductible	250	500	1000
Average claim size	2000	2700	4000

Estimate b using regression.

A. Less than 3.0

B. At least 3.0, but less than 3.5

C. At least 3.5, but less than 4.0

D. At least 4.0, but less than 4.5

E. At least 4.5

Exercises continue on the next page . . .

3. You are given the following sample:

3	6	10	18	23	35	49	62	81	105

$$\sum_{i=1}^{10} x_i = 392$$
$$\sum_{i=1}^{10} x_i^2 = 26{,}054$$

These values are fitted to an Inverse Gaussian distribution using the method of moments.

Determine the estimated probability that an observation will be greater than 50.

A. Less than 0.25
B. At least 0.25, but less than 0.30
C. At least 0.30, but less than 0.35
D. At least 0.35, but less than 0.40
E. At least 0.40

4. You are given a sample of 4 observations:

2	5	20	100

A lognormal distribution is fitted to this data in two ways:

1. The method of moments, yielding parameters μ_1 and σ_1
2. Maximum likelihood, yielding parameters μ_2 and σ_2

Determine $\mu_1 - \mu_2$.

A. Less than 0.1
B. At least 0.1, but less than 0.3
C. At least 0.3, but less than 0.5
D. At least 0.5, but less than 0.7
E. At least 0.7

5. For a set of 3 biased coins, the probability of head is p. The 3 coins are tossed 10 times, with the following results:

Number of heads	Number of times
0	4
1	3
2	2
3	1

Determine the maximum likelihood estimate of p.

A. 1/5 B. 1/4 C. 1/3 D. 2/5 E. 1/2

Exercises continue on the next page . . .

6. You are given a sample of size 4 from a distribution with probability density function

$$f(x) = 2x \qquad 0 \le x \le 1$$

$Y_1, \ldots Y_4$ are the order statistics.

Determine $\Pr(Y_2 > 0.5)$.

A. Less than 0.5
B. At least 0.5, but less than 0.6
C. At least 0.6, but less than 0.7
D. At least 0.7, but less than 0.8
E. At least 0.8

7. For a Normally distributed variable X with $\sigma^2 = 2500$, you test H_0: $\mu = 100$ against H_1: $\mu < 100$ using the sample mean of 30 observations. The test is constructed to have 1% significance.

Determine the power of the test at 70.

A. Less than 0.72
B. At least 0.72, but less than 0.76
C. At least 0.76, but less than 0.80
D. At least 0.80, but less than 0.84
E. At least 0.84

8. The amount of time your trip to work takes is a Normally distributed random variable with mean x minutes and variance 25. You would like to test the hypothesis H_0: $x = 30$ against the alternative H_1: $x > 30$. The test should have 5% significance and 90% power at 35.

Determine the minimum number of trips you will need in order to perform this test.

A. 9 B. 10 C. 11 D. 12 E. 13

9. A Normal random variable is known to have mean 5. For a sample of five observations from the variable, $\sum_{i=1}^{5}(x_i - 5)^2 = 175$.

Construct a 95% confidence interval of the form (a, ∞) for the variance.

Determine a.

A. Less than 12
B. At least 12, but less than 14
C. At least 14, but less than 16
D. At least 16, but less than 18
E. At least 18

10. Mortality rates q_x are twice the rates of the Illustrative Life Table for $x < 100$.

Calculate $1000\,{}_{2|}q_{35}$.

A. Less than 4.53
B. At least 4.53, but less than 4.54
C. At least 4.54, but less than 4.55
D. At least 4.55, but less than 4.56
E. At least 4.56

Exercises continue on the next page . . .

11. You are given that $\mu_x = 0.002x + 0.005$.

Calculate ${}_{5|}q_{20}$.

A. Less than 0.043
B. At least 0.043, but less than 0.045
C. At least 0.045, but less than 0.047
D. At least 0.047, but less than 0.049
E. At least 0.049

12. Three independent probes are sent to Mars. The probability distribution for the amount of time each one will function is given in the following table.

Years (n)	Probability of functioning n years
1	0.9
2	0.8
3	0.5

The time to failure is assumed to be uniformly distributed between integral numbers of years.

Calculate the probability that at least two of the probes function for at least $2\frac{1}{3}$ years.

A. Less than 0.40
B. At least 0.40, but less than 0.55
C. At least 0.55, but less than 0.70
D. At least 0.70, but less than 0.85
E. At least 0.85

13. For three independent lives (x), (y), and (z), you are given

(i) $\mu_x(t) = 0.01,\ t > 0$
(ii) $\mu_y(t) = 0.02,\ t > 0$
(iii) $\mu_z(t) = 0.03,\ t > 0$

Calculate $\mathring{e}_{xyz}$.

A. Less than 18
B. At least 18, but less than 22
C. At least 22, but less than 26
D. At least 26, but less than 30
E. At least 30

Exercises continue on the next page . . .

14. For a fully continuous whole life insurance of 1000 on 2 independent lives (x) and (y), you are given

(i) Benefits are payable at the moment of the second death.
(ii) Premiums are payable while both are alive.
(iii) $\mu_x(t)=0.02$ for all t.
(iv) $\mu_y(t)=0.03$ for all t.
(v) $\delta=0.05$

Determine the annual benefit premium.

A. Less than 13
B. At least 13, but less than 15
C. At least 15, but less than 17
D. At least 17, but less than 19
E. At least 19

15. For a double decrement model with decrements from death (1) and withdrawal (2), you are given:

(i) The following decrement rates for (x):

t	Death $q_{x+t-1}^{(1)}$	Withdrawal $q_{x+t-1}^{(2)}$
1	0.003	0.20
2	a	0.15
3	$2a$	0.10

(ii) ${}_3q_x^{(1)}=0.017985$.

Determine a.

A. Less than 0.0045
B. At least 0.0045, but less than 0.0055
C. At least 0.0055, but less than 0.0065
D. At least 0.0065, but less than 0.0075
E. At least 0.0075

16. You are given:

(i) $\mu=0.01$.
(ii) $b_t=1000(1.06^t)$
(iii) $\delta=0.05$

Calculate the actuarial present value of an insurance paying benefit b_t at the moment of death.

A. Less than 5750
B. At least 5750, but less than 5775
C. At least 5775, but less than 5800
D. At least 5800, but less than 5825
E. At least 5825

Exercises continue on the next page . . .

17. A special 9-year term insurance pays the following benefit at the end of the year of death:

Year of death t	1	2	3	4	5	6	7	8	9
Benefit b_t	1	2	3	4	5	4	3	2	1

You are given the following values for increasing and decreasing term insurances:

n	$(IA)^{1}_{x:\overline{n}\rceil}$	$(DA)^{1}_{x:\overline{n}\rceil}$
4	0.5	0.7
5	0.8	1.0
9	2.3	2.8
10	2.9	3.7

Determine the actuarial present value of the term insurance.

A. Less than 0.9
B. At least 0.9, but less than 1.1
C. At least 1.1, but less than 1.3
D. At least 1.3, but less than 1.5
E. At least 1.5

18. You are given:

(i) A special life annuity on (65) pays 100 at the 66th birthday and once every 2 years thereafter (on the 68th birthday, on the 70th birthday, etc.) Mortality is as follows:

$$q_{65+t} = \begin{cases} 0.02 & t=0 \\ 0.03 & t=1 \\ 0.04 & t=2 \\ 0.05 & t\ge 3 \end{cases}$$

(ii) $i=0.05$

Calculate the actuarial present value of this annuity.

A. Less than 525
B. At least 525, but less than 550
C. At least 550, but less than 575
D. At least 575, but less than 600
E. At least 600

Exercises continue on the next page . . .

19. You are given:

(i) $A_{x:\overline{10}|} = 0.62$
(ii) $A^{\ 1}_{x:\overline{10}|} = 0.03$
(iii) $A_x = 0.15$
(iv) $i = 0.05$

Determine ${}_{10|}\ddot{a}_x$.

A. Less than 8.75
B. At least 8.75, but less than 9.00
C. At least 9.00, but less than 9.25
D. At least 9.25, but less than 9.50
E. At least 9.50

20. You are given that $A_x = 0.4 + 0.01x$ for $x < 60$.

Calculate ${}_{20}V_{30}$.

A. $\frac{1}{4}$ B. $\frac{1}{3}$ C. $\frac{1}{2}$ D. $\frac{2}{3}$ E. $\frac{3}{4}$

21. Each night, your supper has one of four main courses: beef, chicken, fish, or pasta. Your main course is always different from the one of the previous night. Each of the 3 main courses not eaten the previous night are equally likely.

Determine the probability that you will have pasta three nights from now, given that you had beef tonight.

A. 1/4 B. 7/27 C. 5/19 D. 4/15 E. 1/3

22. For an auto insurance policy, drivers may be standard or preferred. Every year, the following are the probabilities of not renewing or moving to a different class:

Current class	Does not renew	Moves to standard class	Moves to preferred class	Stays in current class
Standard	0.1	—	0.4	0.5
Preferred	0.1	0.2	—	0.7

For a driver who is currently preferred, calculate the expected number of transitions between standard and preferred classes over the next 3 renewals, given that the driver renews at least 3 times.

A. Less than 0.60
B. At least 0.60, but less than 0.70
C. At least 0.70, but less than 0.80
D. At least 0.80, but less than 0.90
E. At least 0.90

23. Cars arrive at a toll booth in a Poisson process at the rate of 6 per minute.

Determine the probability that the third car will arrive between 30 and 40 seconds from now.

A. Less than 0.18
B. At least 0.18, but less than 0.21
C. At least 0.21, but less than 0.24
D. At least 0.24, but less than 0.27
E. At least 0.27

Exercises continue on the next page . . .

24. A business receives 50 pieces of mail every day in a Poisson process. One tenth of the mail contains checks. The logarithm of the amount of each check has a normal distribution with parameters $\mu = 3$, $\sigma^2 = 9$.

Determine the average number of checks for amounts greater than 10,000 that the business receives in a seven day week.

A. Less than 0.66

B. At least 0.66, but less than 0.69

C. At least 0.69, but less than 0.75

D. At least 0.75, but less than 0.75

E. At least 0.75

25. ATM withdrawals occur in a Poisson process at varying rates throughout the day, as follows:

11PM–6AM	3 per hour
6AM–8AM	Linearly increasing from 3 per hour to 30 per hour
8AM–5PM	30 per hour
5PM–11PM	Linearly decreasing from 30 per hour to 3 per hour

Withdrawal amounts are uniformly distributed on $(100, 500)$, and are independent of each other and the number of withdrawals.

Using the normal approximation, estimate the amount of money needed to be adequate for all withdrawals for a day 95% of the time.

A. Less than 137,500

B. At least 137,500, but less than 138,000

C. At least 138,000, but less than 138,500

D. At least 138,500, but less than 139,000

E. At least 139,000

Solutions to the above questions begin on page 1057.

Appendix A. Solutions to the Practice Exams

Answer Key for Practice Exam 1

1	C	11	A	21	B
2	A	12	D	22	C
3	A	13	A	23	B
4	D	14	C	24	B
5	C	15	D	25	B
6	D	16	C		
7	D	17	D		
8	A	18	B		
9	C	19	E		
10	A	20	D		

Practice Exam 1

1. [**Lesson 5**] The bias is the expected value of the estimator minus the true value of the parameter.

$$\mathbf{E}[\hat{\theta}^2] = \mathrm{Var}(\hat{\theta}) + \mathbf{E}[\hat{\theta}]^2 = 20 + 4^2 = 36$$

and $\theta^2 = 6^2 = 36$, so

$$\mathrm{bias}_{\hat{\theta}^2}(\theta^2) = 36 - 36 = \boxed{\mathbf{0}} \qquad \textbf{(C)}$$

2. [**Lesson 6**]

$$\sum X_i = 250 + 500 + 1000 = 1750$$
$$\sum Y_i = 2000 + 2700 + 4000 = 8700$$
$$\sum X_i^2 = 250^2 + 500^2 + 1000^2 = 1{,}312{,}500$$
$$\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{3} = 1{,}312{,}500 - \frac{1750^2}{3} = 291{,}666.67$$
$$\sum X_i Y_i = (250)(2000) + (500)(2700) + (1000)(4000) = 5{,}850{,}000$$
$$\sum X_i Y_i - \frac{\sum X_i \sum Y_i}{3} = 5{,}850{,}000 - \frac{(1750)(8700)}{3} = 775{,}000$$
$$b = \frac{X_i Y_i - \frac{\sum X_i \sum Y_i}{3}}{\sum X_i^2 - \frac{\left(\sum X_i\right)^2}{3}} = \frac{775{,}000}{291{,}666.67} = \boxed{\mathbf{2.657143}} \qquad \textbf{(A)}$$

3. [**Lesson 3**] From the tables, $\mathbf{E}[X]=\mu$ and $\mathrm{Var}(X)=\mu^3/\theta$. Then

$$\mu=\bar{x}=\frac{392}{10}=39.2$$
$$\frac{\mu^3}{\theta}=\hat{\sigma}^2=\frac{26{,}054}{10}-39.2^2=1068.76$$
$$\theta=\frac{39.2^3}{1068.76}=56.36091$$

From the tables, with

$$z=\frac{x-\mu}{\mu}=\frac{50-39.2}{39.2}=0.27551$$

and

$$y=\frac{x+\mu}{\mu}=z+2=2.27551$$

we have

$$\begin{aligned}F(50)&=\Phi\left(z\left(\frac{\theta}{x}\right)^{1/2}\right)+\exp\left(\frac{2\theta}{\mu}\right)\Phi\left(-y\left(\frac{\theta}{x}\right)^{1/2}\right)\\&=\Phi\left(0.27551\left(\frac{56.36091}{50}\right)^{1/2}\right)+\exp\left(\frac{2(56.36091)}{39.2}\right)\Phi\left(-2.27551\left(\frac{56.36091}{50}\right)^{1/2}\right)\\&=\Phi(0.293)+\exp(2.87556)\Phi(-2.416)\\&=0.6152+17.7354(0.0078)\\&=0.754\end{aligned}$$

The estimated probability of being above 50 is $1-F(50)=\boxed{\mathbf{0.246}}$. **(A)**

4. [**Lessons 3 and 4**] The maximum likelihood estimator of μ is the average of the logarithms of the observations:

$$\mu_2=\frac{\ln 2+\ln 5+\ln 20+\ln 100}{4}=2.47587$$

For the method of moments estimator, we equate the first 2 moments.

$$e^{\mu_1+\sigma_1^2/2}=\frac{2+5+20+100}{4}=31.75$$
$$e^{2\mu_1+2\sigma_1^2}=\frac{4+25+400+10{,}000}{4}=2607.25$$

Divide the fourth power of the first equation by the second equation.

$$e^{2\mu_1}=\frac{31.75^4}{2607.25}=389.7555$$
$$2\mu_1=\ln 389.7555=5.96552$$
$$\mu_1=\frac{5.96552}{2}=2.98276$$

The difference is $\mu_1-\mu_2=2.98276-2.47587=\boxed{\mathbf{0.50689}}$. **(D)**

5. [**Lesson 4**] For a binomial with fixed $m=3$, maximum likelihood estimates q the same way as the method of moments. For 30 tosses (10 tosses of 3 coins) we have $(1)(3)+(2)(2)+(3)(1)=10$ heads, so $q=\frac{10}{30}=\boxed{\mathbf{\frac{1}{3}}}$. **(C)**

6. [**Lesson 16**] The density function for Y_2, $f_{Y_2}(x)$, is the probability of one sample item being below x ($F_X(x) = x^2$) times the density of the distribution ($f_X(x) = 2x$) times the probability of two ample items being above x, $[1 - F_X(x)]^2 = (1 - x^2)^2$, times an appropriate trinomial coefficient:

$$f_{Y_2}(x) = \frac{4!}{1!1!2!} x^2 (2x)(1 - x^2)^2$$

We integrate this from 0.5 to 1. Note that $\frac{4!}{1!1!2!} = 12$.

$$\begin{aligned}
\Pr(Y_2 > 0.5) &= 24 \int_{0.5}^{1} x^3 (1 - x^2)^2 \mathrm{d}x \\
&= 24 \int_{0.5}^{1} (x^3 - 2x^5 + x^7) \mathrm{d}x \\
&= 24 \left(\frac{1}{4} \left(1 - 0.5^4\right) - \frac{2}{6} \left(1 - 0.5^6\right) + \frac{1}{8} \left(1 - 0.5^8\right) \right) \\
&= 6(0.9375) - 8(0.984375) + 3(0.996094) = \boxed{\mathbf{0.7383}} \qquad \textbf{(D)}
\end{aligned}$$

7. [**Lesson 8**] To achieve 1% significance, the critical value for a normal random variable must be 2.326 times the standard deviation below the mean, or $100 - 2.326(\frac{50}{\sqrt{30}}) = 78.76$. The power of the test at 70 is the probability of rejecting the null hypothesis if $\mu = 70$, or

$$\Pr(X < 70) = \Phi\left(\frac{78.76 - 70}{50/\sqrt{30}}\right) = \Phi(0.960) = \boxed{\mathbf{0.831}} \qquad \textbf{(D)}$$

8. [**Lesson 9**] If the critical value is x and n is the number of trips, we need $x = 30 + 1.645(\frac{5}{\sqrt{n}})$ for the significance condition, and we need $x \le 35 - 1.282(\frac{5}{\sqrt{n}})$ for the power condition. Thus we have

$$\begin{aligned}
\frac{(1.645 + 1.282)(5)}{\sqrt{n}} &= 5 \\
\sqrt{n} &= 2.927 \\
n &= 8.567
\end{aligned}$$

Rounding up to the next integer, $\boxed{9}$ trips are needed. **(A)**

9. [**Lesson 12**] Let X_i be an observation of the normal random variable and σ^2 the variance of X_i. Let $W = \sum_{i=1}^{5} (X_i - 5)^2 / \sigma^2$. Then by the definition of the chi-square distribution, W is a chi-square random variable with 5 degrees of freedom. The observed value of W is $175/sigma^2$, so

$$\sigma^2 \sim \frac{175}{W}$$

To find the lower bound a of a 95% confidence interval, we use the 95th percentile of W, or 11.070:

$$a = \frac{175}{11.070} = \boxed{\mathbf{15.808}} \qquad \textbf{(C)}$$

10. [**Lesson 17**] We have:

x	ILT q_x	q_x
35	0.00201	0.00402
36	0.00214	0.00428
37	0.00228	0.00456

Therefore

$$1000\,{}_{2|}q_{35} = 1000(1-0.00402)(1-0.00428)(0.00456) = \boxed{\mathbf{4.52223}} \qquad \textbf{(A)}$$

11. [**Lesson 18**] ${}_{5|}q_{20} = \big(s(25)-s(26)\big)/s(20)$, so we will calculate these three values of $s(x)$. (Equivalently, one could calculate ${}_5p_{20}$ and ${}_6p_{20}$ and take the difference.) The integral of μ_x is

$$\int_0^x \mu_u \, du = \left(\frac{0.002u^2}{2} + 0.005u\right)\Bigg|_0^x = 0.001x^2 + 0.005x$$

so

$$\begin{aligned}
s(20) &= \exp\Big(-\big(0.001(20^2)+0.005(20)\big)\Big) = \exp(-0.5) = 0.606531\\
s(25) &= \exp\Big(-\big(0.001(25^2)+0.005(25)\big)\Big) = \exp(-0.75) = 0.472367\\
s(26) &= \exp\Big(-\big(0.001(26^2)+0.005(26)\big)\Big) = \exp(-0.806) = 0.446641
\end{aligned}$$

and the answer is

$${}_{5|}q_{20} = \frac{0.472367-0.446641}{0.606531} = \boxed{\mathbf{0.042415}} \qquad \textbf{(A)}$$

12. [**Lesson 22**] The probability of one probe surviving $2\frac{1}{3}$ years is linearly interpolated between 0.8 and 0.5, $0.8-\left(\frac{1}{3}\right)(0.8-0.5) = 0.7$. The probability of at least two surviving is the probability of exactly two surviving (and there are three ways to choose 2 out of 3) or three surviving, or

$$3(0.7^2)(0.3) + 0.7^3 = \boxed{\mathbf{0.784}} \qquad \textbf{(D)}$$

13. [**Lesson 44**]

$$\begin{aligned}
{}_tp_{xyz} &= e^{-0.06t}\\
\mathring{e}_{xyz} &= \int_0^\infty e^{-0.06t}\,dt = \frac{1}{0.06} = \boxed{\mathbf{16.6667}} \qquad \textbf{(A)}
\end{aligned}$$

14. [**Lesson 45**]

$$\begin{aligned}
\bar{A}_{\overline{x:y}} &= \bar{A}_x + \bar{A}_y - \bar{A}_{xy}\\
&= \frac{0.02}{0.07} + \frac{0.03}{0.08} - \frac{0.05}{0.10} = 0.16071\\
\bar{a}_{xy} &= \frac{1}{0.02+0.03+0.05} = 10
\end{aligned}$$

The annual benefit premium is $1000(0.16071/10) = \boxed{\mathbf{16.071}}$. **(C)**

15. [**Lesson 47**] We have ${}_{1|2}q_x^{(1)} = {}_3q_x^{(1)} - q_x^{(1)} = 0.017985 - 0.003 = 0.014985$. Also, $p_x^{(\tau)} = 1 - 0.003 - 0.20 = 0.797$. We set up an equation for ${}_{1|2}q_x^{(1)}$ and solve.

$$\begin{aligned}
{}_{1|}q_x^{(1)} + {}_{2|}q_x^{(1)} &= {}_{1|2}q_x^{(1)} \\
(0.797)(a) + (0.797)(1 - a - 0.15)(2a) &= 0.014985 \\
0.797a + 1.3549a - 1.594a^2 &= 0.014985 \\
1.594a^2 - 2.1519a + 0.014985 &= 0 \\
a = \frac{2.1519 - \sqrt{4.535129}}{3.188} &= \boxed{\mathbf{0.007}} \qquad \textbf{(D)}
\end{aligned}$$

The other solution to the quadratic is rejected since it is greater than 1.

16. [**Lesson 24**] The actuarial present value is

$$\begin{aligned}
\int_0^\infty 1000(1.06^t)e^{-0.06t}0.01\,\mathrm{d}t &= 10\int_0^\infty e^{(\ln 1.06 - 0.06)t}\,\mathrm{d}t \\
&= \frac{10}{0.06 - \ln 1.06} = \boxed{\mathbf{5776.70}} \qquad \textbf{(C)}
\end{aligned}$$

17. [**Lesson 27**] The benefits are a 9-year decreasing insurance minus twice a 4-year decreasing insurance. $2.8 - 2(0.7) = \boxed{\mathbf{1.4}}$. **(D)**

18. [**Lesson 29**] The actuarial present value of the payment at 66 is $100\left(\frac{0.98}{1.05}\right) = 93.33$. The values of the payments at 68 and subsequent follow a geometric series with the first term $100\left(\frac{{}_3p_{65}}{1.05^3}\right) = 100\left(\frac{(0.98)(0.97)(0.96)}{1.05^3}\right) = 78.832$ and ratio $\frac{{}_2p_{68}}{1.05^2} = \left(\frac{0.95}{1.05}\right)^2 = 0.81859$, so the answer is

$$93.333 + \frac{78.832}{1 - 0.81859} = 93.33 + 434.55 = \boxed{\mathbf{527.88}} \qquad \textbf{(B)}$$

19. [**Lesson 29**]

$$\begin{aligned}
\frac{1}{d} &= \frac{1+i}{i} = \frac{1.05}{0.05} = 21 \\
\ddot{a}_x &= \frac{1 - A_x}{d} = (1 - 0.15)(21) = 17.85 \\
\ddot{a}_{x:\overline{10}|} &= \frac{1 - A_{x:\overline{10}|}}{d} = (1 - 0.62)(21) = 7.98 \\
{}_{10|}\ddot{a}_x &= \ddot{a}_x - \ddot{a}_{x:\overline{10}|} = 17.85 - 7.98 = \boxed{\mathbf{9.87}} \qquad \textbf{(E)}
\end{aligned}$$

20. [**Lesson 39**] By the insurance ratio formula (39.2),

$$\begin{aligned}
A_{30} &= 0.4 + 0.01(30) = 0.7 \\
A_{50} &= 0.4 + 0.01(50) = 0.9 \\
{}_{20}V_{30} &= \frac{A_{50} - A_{30}}{1 - A_{30}} = \frac{0.9 - 0.7}{1 - 0.7} = \boxed{\mathbf{\frac{2}{3}}} \qquad \textbf{(D)}
\end{aligned}$$

21. [**Lesson 49**] This is a Markov chain. The transition matrix going in the beef, chicken, fish, pasta order (actually the order doesn't matter) is

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix}$$

The probabilities of fish, chicken, pasta are 1/3 apiece on the first night, making the state vector $(0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Multiplying this by the transition matrix, we get for the second night's state vector

$$\begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{2}{9} & \frac{2}{9} & \frac{2}{9} \end{pmatrix}$$

For the third night's pasta, we multiply the state vector by the last column of the transition matrix (in order to get the last entry of the third state vector):

$$\frac{1}{3}\left(\frac{1}{3}\right) + \frac{2}{9}\left(\frac{1}{3}\right) + \frac{2}{9}\left(\frac{1}{3}\right) + \frac{2}{9}(0) = \boxed{\mathbf{\frac{7}{27}}} \qquad \textbf{(B)}$$

22. [**Lesson 49**] We have a homogeneous Markov chain with three states: preferred, standard, and gone (not renewed). The transition matrix, with the three states in this order, is

$$\begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0 & 0 & 1 \end{pmatrix}$$

Since we have to condition the expectation on renewing 3 times, we consider all 8 possible paths for a driver who renews 3 times, calculate the probability of each path and the number of transitions in each path. Since the probability of renewal is 0.9 each year, the probability of renewing 3 times, which will be the sum of the 8 probabilities of the paths, is $0.9^3 = 0.729$. In the following table, the path PSS (for example) means being in the preferred state after the first renewal and the standard state after the second and third renewals. The probability of each path is the product of the probabilities of the 3 transitions. All probabilities in this table are unconditional; they are not conditioned on renewing.

Path	Probability of Year 1	Probability of Year 2	Probability of Year 3	Probability of Path	Number of Transitions
PPP	0.7	0.7	0.7	0.343	0
PPS	0.7	0.7	0.2	0.098	1
PSP	0.7	0.2	0.4	0.056	2
PSS	0.7	0.2	0.5	0.070	1
SPP	0.2	0.4	0.7	0.056	2
SPS	0.2	0.4	0.2	0.016	3
SSP	0.2	0.5	0.4	0.040	2
SSS	0.2	0.5	0.5	0.050	1
Total				0.729	0.570

0.570 is the weighted average of the probabilities:

$$0.343(0)+0.098(1)+0.056(2)+0.070(1)+0.056(2)+0.016(3)+0.040(2)+0.050(1)=0.570$$

The conditional expected value is 0.0570 divided by 0.729: $0.570/0.729 = \boxed{\mathbf{0.78189}}$. **(C)**

23. [**Lesson 51**] The probability that the third car will arrive in the interval $(30,40)$ is the probability of at least 3 cars in 40 seconds minus the probability of at least 3 cars in 30 seconds. For 40 seconds, the Poisson parameter is 4 and the probability is

$$1-e^{-4}\left(1+4+\frac{4^2}{2}\right)=1-0.238103$$

For 30 seconds, the Poisson parameter is 3 and the probability is

$$1-e^{-3}\left(1+3+\frac{3^2}{2}\right)=1-0.423190$$

The difference is $0.423190-0.238103=\boxed{\mathbf{0.185087}}$. **(B)**

24. [**Lesson 53**] The probability of a check greater than 10,000 is

$$1-\Phi\left(\frac{\ln 10{,}000-3}{3}\right)=1-\Phi(2.07)=1-0.9808=0.0192$$

The Poisson distribution of just the checks over 10,000 in one week has parameter $7(50)(0.1)(0.0192)=\boxed{\mathbf{0.672}}$. **(B)**

25. [**Lesson 55**] The Poisson parameter per day is computed by adding up the rates over the 4 periods. For 11PM–6AM, we have 7 hours times 3 per hour, or 21. For 8AM–5PM we have 9 hours times 30 per hour, or 270. For the other two periods, because of the linear increase or decrease, the average per hour is the midpoint, or $(30+3)/2=16.5$, and there are 8 hours with varying rates, for a total of $8\times 16.5=132$. The total number of withdrawals per day is $21+270+132=423$. The mean aggregate withdrawals is $(423)(300)=126{,}900$.

The second moment of the uniform distribution on $(100,500)$ is the variance plus the mean squared. The variance of a uniform distribution is the range squared divided by 12, or $400^2/12$. Therefore, the second moment of the uniform distribution is $400^2/12+300^2=103{,}333\frac{1}{3}$. The variance of aggregate withdrawals, by the compound variance formula (55.2), is $\lambda\,\mathbf{E}[X^2]=(423)(103{,}333\frac{1}{3})=43{,}710{,}000$.

The amount of money needed to be adequate 95% of the time is

$$126{,}900+1.645\sqrt{43{,}710{,}000}=\boxed{\mathbf{137{,}775.68}}\qquad\textbf{(B)}$$